Matching with Searching*

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Abstract

This paper models a two-sided matching market where workers can only match with firms that they are acquainted with. Each worker may conduct searches to acquaint himself with new firms. Concepts of stability and efficiency are defined. We use the marginal value theorem of linear programming to show workers never search too little in the sense that all stable matchings are efficient. We then prove that in a large market which satisfies some regularity conditions, workers never search too much, i.e. all efficient matchings can be made stable.

1 Introduction

Traditional matching models assume that there is no search friction, so that every agent knows all of his potential partners and can freely match with any one of them. This may be true of matching markets with a small number of participants. However, in decentralized matching among a large number of agents costly searching is often unavoidable. It is unlikely that a man looking for a wife knows all single women in his city, let alone his country. With a limited number of female acquaintances, he either marries a woman he already knows or searches for someone he does not yet know. Similarly, a worker looking for a job often has to search long and hard for a satisfactory position.

This paper considers a two-sided matching market of workers and firms, and relaxes the standard assumption that any worker-firm pair can match. Agents are allowed to match only with those they are acquainted with. Workers but not firms are able to

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conduct searches and enlarge their set of acquainted agents. A natural question to ask is whether workers search in socially optimal ways. This is not obvious because an agent does not take into account other agents that are affected by his search. For example, say that a worker searches for and finds a desirable job opening. If the worker joins the firm, the latter enjoys the surplus of the match without incurring any search effort. Hence there is a reason for workers to search too little unless they can internalize the firms' benefit. On the other hand, if the worker leaves his old job to work at the newly found firm, or the firm discharges incumbent workers to hire the new worker, the displaced agents are likely to be worse off. This implies that excessive searching is also possible.

Our first result is that workers never search too little (no under-search). We then show that when the market is large and satisfies some regularity conditions, workers never search too much (no over-search).

We use the concepts of stability and efficiency to establish these results. A matching is stable if no worker wishes to deviate, either by matching with a different firm he already knows or by conducting searches to find other firms. A matching is efficient if no such deviations can benefit all agents as a whole. If all stable matchings are efficient, by contraposition, whenever there is room for enhancing efficiency at least one agent wishes to deviate. It is in this sense that we claim no under-search. Conversely, we will say that no over-search holds when all efficient matchings can be made stable. This means that when there is no need for further search, we can always find a surplus allocation that leaves every agent satisfied with status quo.

This paper is a natural extension of the classical treatment of assignment games by Shapley and Shubik (1971). To our knowledge, this is the first paper to directly incorporate searching frictions into a matching model and introduce adequate notions of stability and efficiency. A related work is Liu *et al.* (2014), in which agents have incomplete information about others' types.

The job search literature of labor economics Mortensen and Pissarides (1999) considers a similar topic as ours, albeit with exogenous matching or searching rules. Wu (2015) examines whether a random pairwise meeting can lead to stable matchings.

2 The Model

2.1 The Economy

There is a set of workers, I, and a set of firms, J. A match between worker $i \in I$ and firm $j \in J$ produces joint surplus of $v(i, j) \in \mathbb{R}^+ \cup \{0\}$. Utility is fully transferable so that the joint surplus is divided into worker's wage and firm's profit. To this standard matching environment, we add the restriction that a worker and a firm must be *acquainted* in order to match. This is represented by the acquaintance function $q : I \times J \to \{0, 1\}$, where $q(i, j) = 1$ if worker i and firm j are acquainted and $q(i, j) = 0$ if they are not. Worker i's acquaintance set is defined as $Q_q(i) = \{j \mid q(i, j) = 1\}$. An acquainted pair of a worker and a firm may match without cost. Note that classical matching models assumes $q(i, j) = 1$ for all workers i and firms j.

Workers have the option to increase their acquaintance set by searching. Define worker i's search pool as the set of firms yet unknown to him, or $S_q(i) = J - Q_q(i)$. When worker i conducts a search, he randomly finds a firm in his search pool, i.e. the acquaintance function changes to q' where $q'(i, j) = 1$ for some $j \in S_q(i)$ and $q' = q$ everywhere else. Let $f_i(j; q)$ be the probability that i finds j, given acquaintance q, and let $f = (f_i)$. We allow workers to search multiple times until they are satisfied with their acquaintance set. Worker i's k'th search costs him $c(i, k)$. A *matching problem* is defined by the 6-tuple (I, J, v, q, f, c).

Before proceeding, we note some assumptions implicit in our setting. Acquaintance is always mutual in that if a worker knows a firm, the firm knows the worker. The agents know the exact distribution of all agents in the market, but must still make a costly search to actually locate a partner, which is a standard assumption in searching models.

Assumptions on the searching technology are as follows.

- Searching is one-sided, i.e. firms cannot search for workers.
- Searching is independent, i.e. a worker's search does not affect the search of other workers. For example, workers cannot share search results.
- On-the-job search is allowed. A worker can search without giving up his currently matched firm.
- There is only one search pool for each worker. In other words, a worker does not have multiple searching options.

2.1.1 Matching

A matching needs to describe which workers match with which firms, and also how each matched pair divides its surplus. Define the assignment function $a : I \times J \to \{0, 1\}$, where $a(i, j) = 1$ if worker i is matched with firm j and $q(i, j) = 0$ if otherwise. Let $x : I \to \mathbb{R}^+ \cup \{0\}$ and $p : J \to \mathbb{R}^+ \cup \{0\}$ denote workers' and firms' utilities, respectively. These three functions define a matching $M = (a, x, p)$. Let $m(i)$ ($m(j)$) denote the firm(worker) that is matched with worker i (firm j) under matching M.

Assignment a is feasible if for each i, $a(i, j) = 1$ for at most one j, and for each j, $a(i, j) = 1$ for at most 1 i. Matching $M = (q, a, x, p)$ is feasible if assignment a is feasible and satisfies $a \leq q$, and $x(i) + p(j) \leq v(i, j)$ for all pairs (i, j) such that $a(i, j)$=1.

2.1.2 Searching Scheme

I allow workers to search in groups and therefore define *searching schemes*.

Definition 1. *Let q^0 be the initial acquaintance and let $G \subset I$ be a subset of workers. A* ***searching scheme by G*** *is a function $s_G : \mathcal{H} \to G \cup \{0\}$ that specifies which worker will search after each realized history $h = (q^0, (i_1, j_1), (i_2, j_2), \cdots)$ of search results, with $s_G(h^*) = 0$ meaning that search ends at h^*.*

When $G = \{i\}$, we write $s_G = s_i$ and call this **worker i's individual searching scheme**. When we restrict attention to a particular worker and his individual searching scheme, we may write the scheme's history as a string of acquainted firms, $(q^0, j_1, j_2, \cdots)$. Let $s_\emptyset$ denote the searching scheme that does not conduct any search.

2.2 Stability and Efficiency

First recall the standard notion of stability.

Definition 2. *A feasible assignment a is* ***rematch stable*** *if no acquainted worker-firm pair can form a blocking pair, i.e. $x(i) + p(j) \geq v(i, j)$ for all $i \in I$ and $j \in J$ such that $q(i, j) = 1$.*

To this standard definition we add the requirement that no individual worker wishes to conduct a search. It is thus necessary to pin down a worker's benefit from a search scheme. As is the case with rematch stability, we assume that workers see firms' surplus p as if they are posted prices. Whenever worker i finds a new firm j, he is able to match with the firm, pay the firm $p(j)$, and receive $v(i, j) - p(j)$ himself. If worker i searches k times and acquaintance q' is realized, worker i's surplus equals

$$u_i(q') = \sup_{j \in Q_{q'}(i)} \{v(i, j) - p(j)\} - \sum_{t=1}^{k} c(i, t). \tag{1}$$

Each individual searching scheme s_i induces a probability measure on the set of all acquaintances q' that can be reached from q^0, and, equivalently, on the set of worker i's surplus $u_i(q')$ from these acquaintances. Denote worker i's expected surplus from scheme s_i by $\Pi_i(s_i) \equiv \mathbb{E}[u_i(q^k) \,|\, s_i]$, and let $s_i^* \equiv \arg\max_{s_i \neq s_\emptyset} \Pi_i(s_i)$ be i's **optimal individual search scheme.**

Definition 3. *A feasible matching $M = (a, x, p)$ is* ***stable*** *if it is rematch stable and no worker strictly wishes to search, i.e. $\max_{s_i} \Pi_i(s_i) \leq x(i)$ for all $i \in I$.*

Workers are fully myopic because they only look at the posted prices when deciding whether to search. Therefore I do not have to assume that agents have complete information regarding the parameter functions, including v, q, f, or c. All an agent needs to know is his joint surplus with potential partners, his acquaintance set, his searching technology, and the posted price.

Efficiency, unlike stability, is defined for assignments not matchings since the total surplus is determined only by how agents match, and not by how they divide the gains. An assignment is efficient if no rematching among acquainted agents, nor any searching scheme, can increase total surplus.

Definition 4. *A feasible assignment a is* ***rematch-efficient*** *if no rematch among acquainted agents can increase total surplus, i.e. a maximizes $\sum_{i,j} v(i,j)a(i,j)$ among all feasible assignments.*

Define the value $V(q)$ of an acquaintance q as

$$V(q) = \max_{a(i,j)} \{\sum_{i,j} v(i,j)a(i,j) \,|a \text{ is feasible under } q\}. \tag{2}$$

Given an initial acquaintance q, suppose that a search scheme s_G results in a new acquaintance q. Define $B(q_0, q) = V(q) - V(q_0)$, and let $C(q_0, q)$ denote the realized sum of search costs.[1] Then $B(q_0, q) - C(q_0, q)$ represents the net social benefit from moving to q from q_0. Define

$$\Phi(s_G) = \mathbb{E}[B(q_0, q) - C(q_0, q) \,|\, s_G] \tag{3}$$

as the expected value of search scheme s_G.

Definition 5. *A feasible assignment a is* ***efficient*** *if it is rematch efficient and no search scheme can improve efficiency, i.e. $\Phi(s_G) \leq 0$ for any s_G.*

3 Efficiency of Stable Matchings

Our first result is that under-search never happens. Agents never search too little in the sense that if any matching has room to improve efficiency, it is necessarily unstable. In particular, if there is a search scheme that improves efficiency, at least one agent that participates in this search scheme wishes to conduct an individual search.

Proposition 1. *All stable matchings are efficient.*

[1] Note that C is well-defined.

Proof. It is well known from Shapley and Shubik (1971) that the classical matching problem can be understood as a linear program. Our key insight is to express the acquaintance restriction as linear constraints and then view searching as a relaxation of these constraints.[2]

Consider the classical model with no search friction, $q(i,j) = 1$ for all i and j. We can find an efficient assignment a by maximizing the total surplus, subject to the linear constraints that each agent can match with at most one other agent. This is written as follows.

$$\begin{aligned}
& \max_{a(i,j)\in\mathbb{R}} \sum_{i,j} v(i,j)a(i,j) & \\
s.t. \quad & a(i,j) \geq 0 & \forall i,j \\
& \textstyle\sum_j a(i,j) \leq 1 & \forall i \\
& \textstyle\sum_i a(i,j) \leq 1 & \forall j
\end{aligned}$$

The above program does not explicitly account for the fact that $a(i,j)$ can only take values of 0 or 1. However, it is known that because the constraint matrix is **totally unimodular (TUM)**[3] and has integral entries, the linear program has an integral solution.

This paper adds the restriction that only acquainted pairs may match. We then obtain the following linear program, which we call $\mathcal{P}$.

$$\begin{aligned}
& \max_{a(i,j)} \sum_{i,j} v(i,j)a(i,j) & \\
s.t. \quad & a(i,j) \geq 0 & \forall i,j \\
& \textstyle\sum_j a(i,j) \leq 1 & \forall i \\
& \textstyle\sum_i a(i,j) \leq 1 & \forall j \\
& a(i,j) \leq 0 & \forall (i,j) \quad \text{s.t.} \quad q(i,j) = 0
\end{aligned} \tag{$\mathcal{P}$}$$

The above program also has integer solutions. To see why, note that there is another way to incorporate the acquaintance constraint; we may set $v(i,j) = 0$ for any i, j that

[2]This section uses ideas from linear programming, including total unimodularity, duality, and the marginal value theorem. See Vohra (2004) for a treatment of these concepts.

[3]

are unacquainted. In this alternative but equivalent program the constraints are the same as in the classical program without search frictions, so the constraint matrix is still TUM and thus the program has an integer solution. This solution clearly solves $\mathcal{P}$.

Any linear program with a finite optimal solution has a dual program with the same optimized solution. The dual of $\mathcal{P}$, denoted by $\mathcal{D}$, can be written as follows.

$$\begin{aligned}
& \max_{x(i),p(j)} \sum_i x(i) + \sum_j p(j) && && (\mathcal{D}) \\
s.t. \quad & x(i) \geq 0, \quad p(j) \geq 0 && \forall i,j && \\
& x(i) + p(j) \geq v(i,j) && \forall i,j \text{ s.t. } q(i,j) = 1 && \\
& t(i,j) + x(i) + p(j) \geq v(i,j) && \forall i,j \text{ s.t. } q(i,j) = 0 &&
\end{aligned}$$

Note that the constraint for any acquainted pair is the same as the classical stability condition. Also note that because the dual variables $t(i,j)$ are not part of the maximand, the last group of constraints have no bite per se. However, we will use t to characterize the effects of searching.

Assume that a matching $M = (a,x,p)$ is inefficient but rematch-stable. Rematch stability means $x(i) + p(j) \geq v(i,j)$ for any i, j with $q(i,j) = 1$, so with appropriate choices of t, (t,x,p) is a solution to $\mathcal{D}$. In fact, we may choose t_0 in the following way. If $x(i)+p(j) < v(i,j)$, let $t_0(i,j) = v(i,j) - x(i) - p(j)$, and if $x(i)+p(j) \geq v(i,j)$, then let $t_0(i,j) = 0$. Observe that $t_0(i,j)$ is precisely *the benefit to worker i of becoming newly acquainted with firm j, taking the price $p(j)$ as given.*

What, then, is the social benefit from a new acquaintance? When worker i and firm j are newly acquainted, this is equivalent to *relaxing the constraint $a(i,j) \leq 0$ to $a(i,j) \leq 1$.* The marginal value theorem states that the increase in total surplus from this relaxation is weakly smaller than the least possible value of the corresponding dual variable, $t(i,j)$, which solves the dual program $\mathcal{D}$. Since t_0 was a solution to $\mathcal{D}$, the increase in total surplus is weakly smaller than $t_0(i,j)$.

Note that $t_0(i,j)$ was the personal benefit to worker i. Essentially, the personal gain from acquaintance is always weakly greater than the social gain, so whenever a matching has room for surplus increase, workers will take advantage of the opportunity.

Since a was assumed to be inefficient but rematch-stable, there exists a search scheme s_G such that

$$\mathbb{E}[B(q_0,q) \,|\, s_G] > \mathbb{E}[C(q_0,q) \,|\, s_G].$$

A particular realization q is obtained by relaxing some of the acquaintance constraints. Since an agent can match with at most one other agent, only some of these relaxed constraints will actually be consummated.[4] Let $L(q, q_0)$ be the set of (i, j) pairs whose acquaintance and assignment both change from 0 to 1. Then the benefit from moving to q is less than the least value of $\sum_{(i,j)\in L} t(i, j)$ such that t solves $\mathcal{D}$. Denote such t by t^l.

The individual benefit t_0 is clearly larger than t^l. Also, each worker is able to mimic his behavior in s_G to achieve the same linear combinatino of t_0. Therefore at least one worker must strictly wish to search. $\square$

Note that Propostion 1 does not depend on the search technology.

4 Stability of Efficient Matchings

We now consider whether workers may search excessively. We say workers search too much if an efficient assignment does not allow a stable surplus division, since this means that no matter how surplus is divided there is always some worker that wishes to search, even though no searching scheme can increase total surplus.

This section shows that any efficient assignment can be made stable if search cost is non-increasing and the market is large enough that "loops" are unlikely to exist. Though we do not claim that these are necessary conditions, increasing search cost (Appendix A) or existence of loops (Appendix B) may make prevent an efficient assignment from becoming stable. We first define

Definition 6. *For $N \in \mathbb{N}$, an N-**economy** is a matching market (I, J, v, q, f, c) with N workers and N firms, $I = J = \{1, 2, \cdots, N\}$.*

We also define an approximate notion of stability. A pairwise rematch is **ϵ-profitable** for worker i and firm j if $v(i, j) - \epsilon > x(i) + p(j)$. Worker i and firm j are **ϵ-indifferent** to the rematch if $v(i, j) - \epsilon = x(i) + p(j)$. An individual search scheme s_i is **ϵ-profitable** if $\Pi_i(s_i) > \epsilon$, and **ϵ-indifferent** if $\Pi_i(s_i) = \epsilon$.

Definition 7. *Let $\epsilon > 0$. A feasible matching $M = (a, x, p)$ is ϵ-**stable** if no pairwise rematch or individual search scheme is ϵ-profitable.*

Alternatively, we say that a is ϵ-stable under surplus division (x, p). Assume the following regularity conditions hold for a family of economies $\mathcal{F} \subset \{(I, J, v, q, f, c)\}_{N\in\mathbb{N}}$.

- **(R1)** There exists $v_\infty > 0$ such that $v(i, j) \leq v_\infty$ for all $i, j \in \mathbb{N}$.
- **(R2)** For all $i, k \in \mathbb{N}$, $c(i, k) \geq c(i, k + 1)$.
- **(R3)** There exists $c_\infty > 0$ such that $c(i, k) \geq c_\infty$ for all $i, k \in \mathbb{N}$.
- **(R4)** There exists $q_\infty \in \mathbb{N}$ such that $|Q_q(i)| \leq q_\infty$ for all $i \in \mathbb{N}$.

[4] Recall that at any acquaintance, we always have integer solutions to the primal linear program.

(R5) Search probability is uniform, i.e. for any worker i and firm $j, j' \notin Q_i$, $f_i(j; Q_i) = f_i(j'; Q_i)$.

(R6) (No Bunching) Consider a graph with workers as nodes, where workers i_1 and i_2 are connected iff $q(i_1, m(i_2)) = 1$. For any $N \in \mathbb{N}$, the number of workers in any connected component of any N-economy cannot exceed l.

Condition **(R1)** means that joint surplus v is bounded above. **(R2)** and **(R3)** imply that cost c is non-increasing but bounded away from 0. **(R4)** says there is an upper bound on the number of firms that any worker is originally acquainted with.

Intuitively, **(R6)** means we can always divide the workers into small mutually unacquainted groups even as the market grows large. Two workers are connected if and only if one worker is acquainted with the firm that is matched with the other worker. A subset of workers constitute a connected component if all workers in the set are connected, either directly or via a path of connections. The assumption states that as the market grows large, the size of any connected component is bounded. For example, this is trivially satisfied if all workers have zero acquaintance and hence must search for a new firm in order to deviate from a match.

In this section, we restrict attention to $\mathcal{F} \subset \{(I, J, v, q, f, c)\}_{N \in \mathbb{N}}$ that satisfies regularity conditions **(R1)** to **(R6)**.

Recall that in an individual search scheme s_i, at every history h the worker decides whether to continue searching or stop. A firm is satisfactory to a worker if the worker stops searching when he finds the firm. Formally, firm j is **satisfactory for worker i at a non-terminal history** h if $s_i^*(h) = i$ and $s_i^*(h') = 0$, where $h' = (h, (i, j))$. Equivalently, we say that worker i was **successful** at h'. The following lemmas hold in $\mathcal{F}$.

Our first lemma says that satisfactory workers are indeed "satisfactory".

Lemma 1. *Whenever an optimal individual search scheme ends, the worker maximizes his surplus by matching with the firm found at the last search.*

Proof. Assume to the contrary that an optimal individual search scheme has ended with terminal history h, and the worker's surplus is maximized by matching with a firm that was found at $h' < h$. Consider the worker's decision at h'. If he stops, he receives the same surplus as when stopping at h. If he searches, his expected surplus is lower than the expected surplus from searching at h. This is because there is the possibility of finding firms that he is acquainted with at h but not at h', which are "useless" firms because the worker prefers the firm he found at h'. We thus have a contradiction, since if the worker stopped searching at h, he must have already stopped at h'. □

Lemma 1 implies that a worker never "gives up" in the middle of a search to return to a previously acquainted firm. We therefore have the following corollary.

Corollary 1. *If a worker has an optimal individual search scheme, the expected benefit of this scheme does not change when the worker is dismissed by his currently matched firm.*

Lemma 2 can be shown with a logic similar to that of Lemma 1.

Lemma 2. *In any optimal individual search scheme, the set of satisfactory firms weakly decreases, in set inclusion, as the search scheme progresses.*

Proof. Assume otherwise. Then for some economy, and some worker i and firm j in this economy, there exists history h resulting from i's optimal individual search scheme such that j is unsatisfactory to i at h, but satisfactory at $h' = (h, j')$ for some $j' \neq j$. This implies three things about i's searching decision; he searches at both (h, j) and (h, j'), but not at (h, j', j). Worker i's wage at (h, j', j)[5] must equal his wage at either (h, j) or (h, j'). Assume without loss of generality that i 's wage is the same in (h, j', j) and (h, j), which means that acquaintance with j' brings no extra benefit. But then, searching at (h, j', j) must be more profitable than searching at (h, j) because at (h, j), there is a possibility of finding j', a firm without any worth. Hence i must wish to search even at (h, j', j), a contradiction. $\square$

The next lemma says that after any non-terminal history in any optimal individual searching scheme, there exists a family-wide lower bound on the proportion of satisfactory firms.

Lemma 3. *There exists $k \in (0, 1]$ such that, for any non-terminal history in any optimal individual scheme in any economy in $\mathcal{F}$, the proportion of satisfactory workers in the searching pool is greater than k.*

Proof. Assume for the moment that the set of unsatisfactory firms stays the same throughout an optimal individual search scheme, and that the number of unsatisfactory firms is F. Then the number of unsatisfactory firms found before the search terminates, which equals the number of searches conducted minus one, follows the negative hypergeometric distribution. Hence the expected number of searches in the scheme equals $\frac{F}{N-F+1} + 1 = \frac{F/N}{1-F/N+1/N} + 1$. In general, since the set of satisfactory firms weakly decreases (Lemma 2), if there are $F(h)$ unsatisfactory firms at a non-terminal history h, then the expected number of future searches is bounded below by $\frac{F(h)/N}{1-F(h)/N+1/N} + 1$. This means the expected search cost cannot be lower than $\left(\frac{F(h)/N}{1-F(h)/N+1/N} + 1\right)c_\infty$. Assume that $F(h)/N$

[5] As explained in Section 2.2., this is calculated as $\sup_{j \in Q_q(i)}\{v(i, j) - p(j)\}$, where q is the acquaintance obtained after (h, j', j).

can become arbitrarily close to 1. As both $F(h)$ and N are positive integers, $N \to \infty$ as $F(h)/N \to 1$. Thus the expected cost becomes arbitrarily large, and since the value of a search scheme cannot exceed v_∞, at some history where expected cost is large enough, it is better for the worker to stop searching. This contradicts our assumption that h was non-terminal. Therefore, the proportion of unsatisfactory workers at a non-terminal history must be bounded away from 1, which means the proportion of satisfactory workers at a non-terminal history is bounded away from 0. $\square$

Lemma 3 says that workers never search when the probability of finding attractive firms is sufficiently low. This may seem incompatible with the so called "limited acceptability" condition, widely assumed in the literature (Kojima (2015)). If, as the condition states, agents find as acceptable only a limited number of partners even as the market grows large, then Lemma 3 precludes any searching in a large enough market. However, the empirical observations of limited acceptability may well be driven by the fact that agents' acquaintance is bounded. Even though there are many partners that an agent would potentially be willing to match, he is only able to report a few of them as acceptable because he does not know who they are.

We are now ready to prove our second proposition.

Proposition 2 (No Over-Search). *Let $\mathcal{F}$ be a family of economies satisfying* (R1) - (R6). *Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any economy in $\mathcal{F}$ that is larger than N, any efficient assignment can be made ϵ-stable by an appropriate choice of wage vector x.*

To prove Proposition 2, we define a procedure, or an algorithm, which we call $\mathcal{A}$. Given an assignment a, $\mathcal{A}$ starts from all workers receiving 0 surplus, $x^0 = 0$. Then we raise the surplus of each worker who wishes to deviate, just until he is ϵ-indifferent, to get x^1. Again, increase the surplus of any worker who wishes to deviate from x^1 to obtain x^2. Repeating this leads to one out of three outcomes. First, x^t may become ϵ-stable at some t.[6] Second, $\mathcal{A}$ may go on forever, i.e. $< x_t >$ is an infinite sequence. In this case, we show that the limit of the sequence x^∞ is ϵ-stable. Third, $\mathcal{A}$ may fail at some t, which can happen when a worker's incentive to deviate is so large that it is impossible to make him indifferent. This actually implies that a can never be made ϵ-stable, regardless of how surplus is divided. However, we show that in any large enough market, a is necessarily inefficient whenever $\mathcal{A}$ fails. Therefore, in the large market, $\mathcal{A}$ gives an ϵ-stable surplus division for any efficient assignment. Below is a formal proof.

Proof of Proposition 2. Procedure $\mathcal{A}$ works as follows. Given an economy (I, J, v, q, f, c) and an assignment a, start with zero wage for all workers, $x^0 = 0$. Note that specifying x

[6] Since a is given, we say that a wage vector x is ϵ-stable to mean that the matching (a, x, p) is ϵ-stable.

also pins down p. Under this surplus division, a worker wishes to deviate by rematching or searching when his wage is low and the price of firms is low. For any worker i that has an ϵ-profitable deviation under $x^0 = 0$, increase his wage just until he is ϵ-indifferent to deviating, under his new wage and the original price vector he faced, p^0_{-i}. Replacing the components of x^0 with the increased wages gives us a new surplus division x^1.

Of course, x^1 is not necessarily stable since increasing a worker's wage, or decreasing the corresponding firm's price, always makes other workers weakly more prone to deviation. In general, $\mathcal{A}$ constructs x^{t+1} from x^t in the following way. For any worker i that has an ϵ-profitable deviation under x^t, let his new wage $x^{t+1}(i)$ be such that he is ϵ-indifferent to deviation under $x^{t+1}(i)$ and p^t_{-i}.

As we follow $\mathcal{A}$ to recursively define x^t, one of three things can occur.

1. At some x^t, no worker has an ϵ-profitable deviation.

Then we immediately have an ϵ-stable matching.

2. The algorithm goes on forever.

Consider the infinite sequence $< x^t >$ and its limit x^∞. This limit exists since x^t is component-wise increasing and bounded above by the joint surplus $v(i, m(i))$. Then the matching $M(q, a, x^\infty, p^\infty)$ is ϵ-stable. To see why, assume that some worker can deviate from x^∞ and benefit by an amount strictly greater than ϵ. Consider a wage vector x^T in the sequence that is arbitrarily close to the limit.[7] For this wage, all workers either do not have an ϵ-profitable deviation or can be made ϵ-indifferent by an arbitrarily small increase in their surplus, which means their gain from deviation is arbitrarily close to ϵ. At the same time, x^T is arbitrarily close to x^∞, so for any worker, the difference between the deviation profits in the two vectors must be arbitrarily small. This contradicts the assumption.

3. For some x^t, it is impossible to construct x^{t+1}.

This happens when at some step t of procedure $\mathcal{A}$, a worker wishing to deviate cannot be made ϵ-indifferent even if he is given all of the joint surplus with his matched firm.[8] In this case, we can find a search scheme that increases expected total surplus and conclude

[7] Formally, if some worker can deviate from x^∞ to gain $\epsilon + \delta$, consider a wage vector where each component is lesser than x^∞ by at most $\frac{\delta}{3}$.

[8] Although not strictly relevant to our proof, a stable matching cannot exist in this case because each step of the procedure is a necessary condition for stability.

that a is inefficient.

Pick some worker that has an ϵ-profitable deviation even if he receives all of the joint surplus with his matched firm, and call him worker 1. Modify x_t by actually giving worker 1 all of the joint surplus and call this allocation $\bar{x}_t$. We consider a deviation scheme $\mathcal{S}$ starting from $M = (a, \bar{x}_t)$.[9]

Intuitively, a deviation scheme is a concatenated chain of optimal individual search schemes and rematchings, along with a surplus allocation rule. After each worker finds a satisfactory firm, the pair matches and the displaced worker conducts his optimal deviation. The scheme ends when no worker is displaced, a displaced worker wishes to stay unmatched, or a single firm is found twice, i.e. a loop occurs.

Formally, a history $\eta \in \mathbb{H}$ is a finite sequence of worker-firm pairs, $< (i_1, j_1), \cdots, (i_n, j_n) >$. Denote the initial history by η_0. Given a non-initial history $\eta \neq \eta_0$, write $\eta - 1$ to denote the immediate predecessor of η. Write $\eta' \prec \eta$ to mean η' precedes η, immediately or otherwise. Let (i_η, j_η) be the last component of history η.[10] Worker i's sub-history η_i lists, in identical order, the terms in η that contain worker i. Recall that s_i^* is the optimal individual search scheme of worker i given $M = (a, \bar{x}_t)$.

Our deviation scheme $\mathcal{S}$ is an algorithm that consists of a function $d : \mathbb{H} \to I$ and two other functions, $a_\eta : I \times J \to \{0, 1\}$ and $x_\eta : I \to \mathbb{R}^+ \cup \{0\}$. Each history η results in a new assignment a_η and a new surplus division x_η. The algorithm is defined as

1. $d(\eta_0) = 1$.

2. For $d(\eta) = i$,

   (a) If $x_\eta(i) \neq 0$ and $v(i,j) - p(j) > x_\eta(i) + \epsilon$ for some $j \in Q_{q_\eta}(i)$, set $\eta \leftarrow (\eta, (i,j))$ and say that **a rematch occurs** at η.

   (b) If $x_\eta(i) \neq 0$ and $v(i,j) - p(j) \leq x_\eta(i) + \epsilon$ for all $j \in Q_{q_\eta}(i)$, worker i conducts a search. When firm j is found, set $\eta \leftarrow (\eta, (i,j))$.

3. If a rematch did not occur at η, and j_η was unsatisfactory for i_η at $\eta - 1$, then take $d(\eta) = i_\eta$, $a_\eta = a_{\eta-1}$, and $x_\eta = x_{\eta-1}$. Return to 2-(b).

4. If a rematch occurred in 2-(a) or a satisfactory worker was found in 2-(b), consider j_η. If there exists $\eta' \prec \eta$ such that $j_{\eta'-1} = j_\eta$ and $i_{\eta'-1} \neq i_{\eta'}$, say that **a loop occurs** at η. Take $a_\eta = a_{\eta-1}$ and $x_\eta = x_{\eta-1}$. End algorithm.

[9]In principle, we need only look at changes in the assignment a to determine changes in total surplus. Hence we only need to find a search scheme to show the inefficiency of a. However, also considering the initial surplus allocation $\bar{x}_t$ simplifies the argument, so we introduce the idea of a deviation scheme which specifies surplus changes for individual agents.

[10]For any $\eta \neq \eta_0$, we have $\eta = (\eta - 1, (i_\eta, j_\eta))$.

5. For $\eta \neq \eta_0$, if a rematch occurred at $\eta - 1$ or if j_η was satisfactory for i_η at $\eta - 1$, and no loop occurred at η, then make the following updates.

   (a) $a_\eta(i_\eta, j_\eta) = 1$. If $m(j_\eta) \neq 0$, then $a_\eta(m(j_\eta), m(m(j_\eta))) = 0$. Otherwise, $a_\eta = a_{\eta-1}$.

   (b) $x_\eta(i_\eta) = v(i_\eta, j_\eta) - p(j_\eta)$ and $x_\eta(m(j_\eta)) = 0$. Otherwise, $x_\eta = x_{\eta-1}$.

   If $m(j_\eta) = 0$, or $m(j_\eta) \neq 0$ and $x(m(j_\eta)) = 0$, end algorithm. If $m(j_\eta) \neq 0$ and $x(m(j_\eta)) \neq 0$, take $d(\eta) = m(j_\eta)$ and return to 2-(a).

This algorithm induces a random terminal history, where the randomness is from the searching technology. Denote the realized terminal history by η_{ter}. We claim that the above deviation scheme is welfare-improving. Precisely, the assignment that we obtain when the algorithm ends gives a larger total surplus than the initial assignment a, in expectation, after taking into account the search costs. To show this, we compare $M = (a, \bar{x}_t)$ with $M_{\eta_{ter}} = (a_{\eta_{ter}}, x_{\eta_{ter}})$.

First note that all firms are indifferent because their surplus never changes even when they match with different workers. In particular, worker 1's original partner $m(1)$ is at least indifferent as he was receiving 0 in M.

Second, consider workers who take actions in the algorithm. Because of how we have constructed M, any worker that is receiving a positive wage in M is ϵ-indifferent and can thus increase his expected surplus by at least ϵ, by either rematching or conducting his optimal individual search. Moreover, this statement holds even after the said worker loses his currently matched firm because, according to Corollary 1, the expected surplus from a profitable search scheme does not depend on a worker's current match.

It is therefore clear that the deviation scheme increases expected surplus *as long as all workers match with their new partners*. This is true when the algorithm ends without a loop, but when there is a loop the two workers whose new partner firm coincides cannot both match with the firm. Our algorithm specifies that the worker who came first matches with the firm, while the second worker is left unmatched. However, the following lemma shows that the probability of the algorithm ending in a loop is sufficiently small in a large market.

Assume that a loop occurs at η_{ter}. Let η be the last history preceding η_{ter} such that a search occurs at $\eta - 1$, with $\eta = \eta_0$ if no search occurs. In other words, $d(\eta - 1)$ searches, and $d(\eta')$ searches in the algorithm only if $\eta' \preceq \eta - 1$. Intuitively, η is the last history that is brought about by a search. Let m be the number of different workers who appear up to η:

$$m = \left|\{i | i = i_{\eta'} \text{ for some } \eta' \preceq \eta\}\right| \tag{4}$$

We say that the loop occurred for the m-th worker, as even if there were other workers afterwards in the deviation scheme, they only rematched and the loop was inevitable once the m-th worker made his search. We may assume that $m \geq 2$.[11]

Lemma 4 (Loops)**.** *Let $\mathcal{F}$ be a family of economies satisfying* (R1) - (R6)*. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ and a constant $\delta > 0$ such that for any economy larger than N, given any initial assignment a that leads to* (4.2)*, the probability of a loop occurring for the m-th worker of the deviation scheme is less than $\delta(m-2)$ for $m \geq 2$.*

Proof. Consider a $N + q_\infty$ economy, and send $N \to \infty$. Let k be the upper bound on the proportion of unsatisfactory firms in any optimal individual search. Then the expected loss from a loop that occurs for the m-th worker is less than

$$L_N(m) = v_\infty \Big[\frac{l(m-2)}{N} + k\frac{l(m-2)}{N-1} + k^2 \frac{l(m-2)}{N-2} + \cdots \Big] \tag{5}$$

which, for given m, goes to 0 as $N \to \infty$. For ϵ that we used to define ϵ-stability, take N large enough such that $L_N(3) < \epsilon$. Then $L_N(m) < (m-2)\epsilon$ for all $m \geq 3$.

□

As the deviation scheme progresses, a searching worker becomes more likely to encounter a loop because there are more firms that have been already switched partners. The lemma says that this increase is linearly bounded. This is enough because the expected increase in total surplus increases linearly. When a loop occurs at the m-th worker, the expected gain from the deviation scheme is at least $\epsilon(m-1)$, since all but the last worker of the deviation scheme, in which at least m workers participate, benefit by at least ϵ.

To sum up, we have shown that if an assignment a is efficient, the above procedure results in either case **1** or **2**. Hence we can always find a stable matching. □

5 Conclusion

This paper establishes two results for a matching market with search friction. Stability always implies efficiency, while efficient matchings can be made stable in a large market with regularity assumptions.

[11] If a $m = 0$ or $m = 1$, we are naturally given a rematching that increases total surplus.

Appendix A: Over-Search Due to Increasing Cost

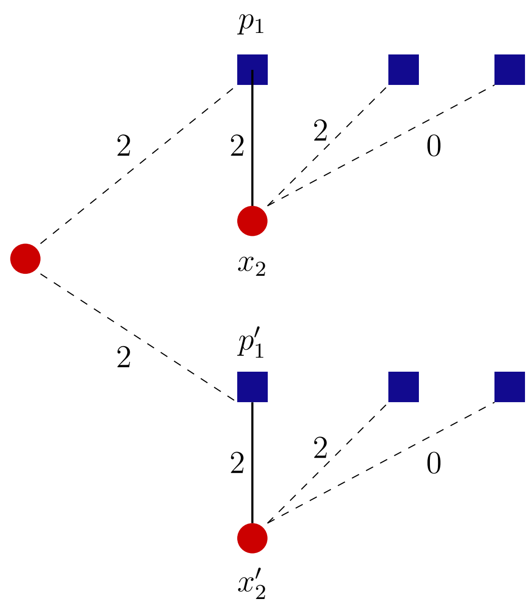

Figure 1

We now demonstrate that agents may search when they should not. Consider a matching market that consists of three workers and six firms. Figure 1 depicts the agents, their acquaintances and searching technologies, and joint surplus values. Workers are represented by circles and firms by squares. The joint surplus from each worker-firm match is depicted next to the line that connects the pair. A solid line between a worker and a firm means that they are already acquainted with each other. Dotted lines represent the workers' searching technology; when a worker makes a search, he gains knowledge of one firm, uniformly chosen among those that are connected to him via dotted lines. For example, if worker 2, currently matched with firm 1, makes a search, then he will find firm 2 with probability 1/2 and firm 3 with probability 1/2. Note that the market is symmetric around worker 1.

We also assume that searching more than once is prohibitively expensive, so workers either search once or do not search at all. Denote worker i's search cost by c_i, and assume $c'_2 = c_2$.

Without any searching, the unique rematch stable matching is one where the two acquainted pairs match, i.e. worker 2 matches with firm 2 and worker 2' matches with firm 2'. Let the surplus division be given by $x(i) = x_i$ and $p(j) = p_j$.

With appropriate choice of c_1 and c_2, we can make the matching efficient but unstable. Efficiency requires that

$$\frac{1}{2}(4 - c_1 - c_2) + \frac{1}{2}(6 - c_1 - c_2) \leq 4,$$

which can be rearranged as

$$c_1 + c_2 \geq 1. \tag{6}$$

On the other hand, conditions for stability are

$$\begin{aligned}
\frac{1}{2}(2-p_1) + \frac{1}{2}(2-p_1') - c_1 &\leq 0 \\
\frac{1}{2}\cdot 2 + \frac{1}{2}x_2 - c_2 &\leq x_2 \\
\frac{1}{2}\cdot 2 + \frac{1}{2}x_2' - c_2 &\leq x_2',
\end{aligned}$$

which implies

$$c_1 + 2c_2 \geq 2. \tag{7}$$

It is clear that equation (4) does not imply (5). For instance, for $c_1 = 1$ and $c_2 = \frac{1}{2} - \epsilon$ efficiency is satisfied by stability is not. For these values of searching cost, even though the initial matching is efficient, there is no way to stop all workers from making searches.

Two things deserve attention in the above example. First, there is no natural search order. It would be desirable in terms of efficiency if worker 1 could search after worker 2 or 2' has been successful in searching, and worker 2 and 2' could search after being located by worker 1. But these two are mutually exclusive, and thus the optimal (non-zero) search is not good enough. This is why the efficiency condition is not strong enough to imply stability.

Second, search cost is increasing. In fact, it is increasing quite rapidly that all workers only consider searching once. At this moment it may not be clear to the reader how this property derives the results. We will show in the next section that even if there is no natural searching order, there is a way to achieve stability if search cost is non-increasing.

Appendix B: Over-Search Due to Loops

References

Kojima, F. (2015). Recent developments in matching theory and its practical applications.

Liu, Q., Mailath, G. J., Postlewaite, A. and Samuelson, L. (2014). Stable matching with incomplete information. *Econometrica*, **82** (2), 541–587.

Mortensen, D. T. and Pissarides, C. A. (1999). New developments in models of search in the labor market. *Handbook of labor economics 3*, pp. 2567–2627.

Shapley, L. S. and Shubik, M. (1971). The assignment game i: The core. *International Journal of Game Theory*, **1.1**, 111–130.

Vohra, R. V. (2004). *Advanced mathematical economics*. Routledge.

Wu, Q. (2015). A finite decentralized marriage market with bilateral search. *Journal of Economic Theory*, **160**, 216–242.